

Darboux integrable system with a triple point and Pseudo-Abelian integrals

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November 15, 2016

Abstract

In this paper we consider the degeneracies of the third type. More exact, the perturbations of the Darboux integrable foliation with a triple point, i.e. the case where three of the curves $\{P_i = 0\}$ meet at one point, are considered. Assuming that this is the only non-genericity, we prove that the number of zeros of the corresponding pseudo-abelian integrals is bounded uniformly for close Darboux integrable foliations. Let \mathcal{F} denote the foliation with triple point (assume it to be at the origin), and let $\mathcal{F}_\lambda = \{M_\lambda \frac{dH_\lambda}{H_\lambda} = 0\}$, M_λ is a integrating factor, be the close foliation. The main problem is that \mathcal{F}_λ can have a small nest of cycles which shrinks to the origin as $\lambda \rightarrow 0$. A particular case of this situation, namely $H_\lambda = (x - \lambda)^\epsilon (y - x)^{\epsilon_+} (y + x)^{\epsilon_-} \Delta$ with Δ non-vanishing at the origin (and generic in appropriate sense).

Mathematics subject classification: 34C07, 34C08

Keywords: Abelian integrals, Limit cycles, Integrable systems

1 Introduction and Main Result

This paper represent an amelioration of [Braghtha] in the sense to eliminate the technic condition on the perturbative one-form η . Precisely, we consider an unfolding ω_λ of the one-form ω_0 , where λ is a small parameter and ω_λ is a family of meromorphic one-forms

$$\omega_\lambda = M_\lambda \frac{dH_\lambda}{H_\lambda}, \quad H_\lambda = P_\lambda^\epsilon \prod_{i=1}^k P_i^{\epsilon_i}, \quad M_\lambda = P_\lambda \prod_{i=1}^k P_i \quad (1)$$

with $\epsilon, \epsilon_i > 0, P_0, P_\lambda, P_i \in \mathbb{R}[x, y]$.

We assume that $P_0(0, 0) = P_1(0, 0) = P_2(0, 0) = 0$ and $P_i(0, 0) \neq 0, i = 3, \dots, k$. Generically, the triple point unfolds into three saddles p_0^λ, p_1^λ and p_2^λ correspond to the transversal intersections of level curves $P_1^{-1}(0)$ and $P_\lambda^{-1}(0)$, $P_1^{-1}(0)$ and $P_2^{-1}(0)$, and $P_2^{-1}(0)$ and $P_\lambda^{-1}(0)$. Here also appears a center p_c^λ in the triangular region bounded by these levels curves.

Consider a polynomial perturbation $\omega_{\lambda, \kappa} = \omega_\lambda + \kappa\eta$, $\kappa > 0$ of the system $\omega_\lambda = M_\lambda \frac{dH_\lambda}{H_\lambda}$, where

$$\eta = Rdx + Sdy,$$

and $R, S \in \mathbb{R}[x, y]$ are a polynomials of degree n . The foliation $\omega_\lambda = 0$ has a maximal nest of cycles $\gamma(\lambda, h) \subseteq \{H_\lambda(x, y) = h\}, h \in (0, n(\lambda))$ filling a connected component of $\mathbb{R}^2 \setminus \{P_\lambda \prod_{i=1}^k P_i = 0\}$, which we denote $D_{(\lambda, h)}$, whose boundary is a polycycle $\gamma(\lambda, 0)$.

To such perturbation one can associate the pseudo-Abelian integral

$$I(\lambda, h) = \int_{\gamma(\lambda, h)} \frac{\eta}{M_\lambda}, \quad (2)$$

which is the principal part of the Poincaré displacement function

$$D(\kappa, \lambda, h) = \kappa h \int_{\gamma(\lambda, h)} \frac{\eta}{M_\lambda} + O(\kappa)$$

of the perturbation $\omega_{\lambda, \kappa}$ along $\gamma(\lambda, h)$.

Let us impose the following generic assumptions:

A₁ : $\frac{\partial P_\lambda}{\partial \lambda}|_{(0,0,0)} \neq 0$.

A₂ : $P_1^{-1}(0), P_2^{-1}(0)$ and $P_0^{-1}(0)$ intersect transversally two by two at the origin which is the only triple point. The level curves $P_i^{-1}(0), i = 3, \dots, k$ intersect transversally and two by two.

Theorem 1. *Let $I(\lambda, h)$ be the family of pseudo-Abelian integrals as defined above. Under assumptions **A₁** and **A₂**, there exists a bound for the number of isolated zeros of $I(\lambda, h) = \int_{\gamma(\lambda, h)} \frac{\eta}{M_\lambda}$. The bound depends only on $n_i = \deg P_i, n = \max(\deg R, \deg S)$ and is uniform in the coefficients of the polynomials P_λ, P_i, R, S , the exponents $\epsilon, \epsilon_i, i = 1, \dots, k$ and the parameter λ .*

1.1 Bobieński result

We recall also a similar result of Bobieński [1] which he prove the existence of a local upper bound for the number of zeros of pseudo-abelian integrals. He consider a one-parameter unfolding of the singular codimension one case. The difference relies in the fact that in this work the Darboux first integrals is more general ($\epsilon_1 \neq \epsilon_2$) and the proof of our main result is purely geometric: we use the blow-up in families. This approach gives directly uniform validity of our study of the pseudo-abelian integrals.

2 Darboux integrable foliation

Let us establish a local normal form near the triple point $(0, 0, 0)$ for the unfolding of the degenerate polycycle H_0 .

Proposition 1. *Under above assumptions **A₁**, **A₂**. There exists a local analytic coordinate system (x, y, λ) at $(0, 0, 0)$ such that H_λ takes the form*

$$H_\lambda = (x - \lambda)^\epsilon (y - x)^{\epsilon_+} (y + x)^{\epsilon_-} \Delta, \quad \lambda > 0 \quad (3)$$

where Δ is an analytic unity function $\Delta(0, 0, 0) \neq 0$.

Let $\mathcal{F}_1, \mathcal{F}_2$ are two foliations of dimension two in space of the total complex space \mathbb{C}^3 with coordinates (x, y, λ) such that

$$\mathcal{F}_1 : \{H(x, y, \lambda) = P_\lambda^\epsilon \prod_{i=1}^k P_i^{\epsilon_i} = h\}, \quad \mathcal{F}_2 : \{\lambda = \text{constant}\}.$$

Consider the Darboux foliation $\mathcal{F} := \{\mathcal{F}_1, \mathcal{F}_2\}$ of dimension one in \mathbb{C}^3 with coordinates (x, y, λ) which is given by the intersection of the leaves of \mathcal{F}_1 and \mathcal{F}_2 . This foliation has a non-elementary singular point at the origin $(0, 0, 0)$. This foliation has a complicated singularity at the origin $(0, 0, 0)$. To reduce this singularity, we perform a directional blowing-up in the family ω_λ . Note that blow-up in a family was introduced in [5], see also [6].

2.1 Directional Blow-up

The blow-up of \mathbb{C}^3 at the origin is defined as the incidence three dimensional manifold $W = \{(p, q) \in \mathbb{CP}^2 \times \mathbb{C}^3 : q \in p\}$. The blow down $\sigma : W \rightarrow \mathbb{C}^3$ is just the restriction to W of the projection $\mathbb{CP}^2 \times \mathbb{C}^3$. The inverse map $\sigma^{-1} : \mathbb{C}^3 \rightarrow W$ is called blow-up and $\sigma^{-1}(0) = \mathbb{CP}^2$ is called exceptional divisor. The projective space \mathbb{CP}^2 is covered by three canonical charts: $W_1 = \{x \neq 0\}$ with coordinates (v_1, w_1) , $W_2 = \{y \neq 0\}$ with coordinates (u_2, w_2) and $W_3 = \{\lambda \neq 0\}$ with coordinates (u_3, v_3) . W_1, W_2 and W_3 define canonical charts on W , with coordinates (u_1, v_1, w_1) , (u_2, v_2, w_2) and (u_3, v_3, w_3) respectively. The blow-up σ^{-1} is written as:

$$\sigma_1^{-1} = \sigma^{-1}|_{W_1} : x = u_1 \quad y = u_1 w_1 \quad \lambda = w_1 u_1 \quad (4)$$

$$\sigma_2^{-1} = \sigma^{-1}|_{W_2} : x = u_2 v_2 \quad y = v_2 \quad \lambda = w_2 v_2 \quad (5)$$

$$\sigma_3^{-1} = \sigma^{-1}|_{W_3} : x = u_3 w_3 \quad y = v_3 w_3 \quad \lambda = w_3 \quad (6)$$

Let $\sigma^{-1}\mathcal{F}$ be the lift of the foliation \mathcal{F} to the complement of the exceptional divisor \mathbb{CP}^2 . This foliation is regular outside of the preimage of the hypersurface $\{H_\lambda = 0, \lambda = 0\}$.

Proposition 2. *Let $a = \epsilon + \epsilon_- + \epsilon_+$*

1. *The foliation $\sigma^{-1}\mathcal{F}$ extends in a unique way to a holomorphic singular foliation $\sigma^*\mathcal{F}$ on W which we call the blow-up of the original codimension two foliation \mathcal{F} by the map σ .*

2. *Let $\sigma_1^*\mathcal{F}$ be the restriction of the blown-up foliation $\sigma^*\mathcal{F}$ to the chart W_1 . The singularities of $\sigma_1^*\mathcal{F}$ are located at the points $p_+ = (0, 1, 0), p_- = (0, -1, 0), q_+ = (0, 1, 1)$ and $q_- = (0, -1, 1)$. All these singular points are linearisable saddles, with eigenvalues $\mu_+ = (\epsilon_+, -a, -\epsilon_-), \mu_- = (-\epsilon_-, a, \epsilon_-), \nu_+ = (0, -\epsilon, \epsilon_+)$ and $\nu_- = (0, -\epsilon, \epsilon_-)$ respectively.*

Proof. We prove the second item of Proposition. Since $\sigma : W \rightarrow \mathbb{C}^3$ is a biholomorphism outside \mathbb{CP}^2 , all singularities of $\sigma_1^*\tilde{\mathcal{F}}$ on $W_1 \setminus \{u_1 = 0\}$ correspond to singularities of \mathcal{F} .

Thus, it suffices to compute the singularities of $\sigma_1^*\mathcal{F}$ on the exceptional divisor $\{u_1 = 0\}$. On the exceptional divisor, the foliation is given by the levels of

$$G := \frac{(\pi \circ \sigma_1)^a}{H \circ \sigma_1} = w_1^a (1 - w_1)^{-\epsilon} (v_1 - 1)^{-\epsilon_+} (v_1 + 1)^{-\epsilon_-} \tilde{\Delta}^{-1},$$

where $\tilde{\Delta}$ is unit of the form $\tilde{\Delta} = c + u_1 f$, f is a holomorphic function. Let us compute the eigenvalues at p_+, p_-, q_+ and q_- . Near the exceptional divisor $\{u_1 = 0\}$, the foliation $\sigma_1^*\mathcal{F}$ is given by

$$\begin{aligned} H \circ \sigma_1(u_1, v_1, w_1) &= u_1^a (1 - w_1)^\epsilon (v_1 - 1)^{\epsilon_+} (v_1 + 1)^{\epsilon_-} \tilde{\Delta} = h, \\ \pi \circ \sigma_1(u_1, v_1, w_1) &= u_1 w_1 = \lambda. \end{aligned}$$

Near p_+ and after the respective changes of variable $v = (v_1 - 1)(v_1 + 1)^{\frac{\epsilon_-}{\epsilon_+}} \Delta$, the blown-up foliation $\sigma_1^*\mathcal{F}$ is given by the two first integrals $u_1^a v_+^{\epsilon_+} = h$ and $u_1 w_1 = s$. Then, near this point the vector field generating the foliation $\sigma_1^*\mathcal{F}$ is given by

$$X(u_1, v, w_1) = \mu_1^+ u_1 \frac{\partial}{\partial u_1} + \mu_2^+ v \frac{\partial}{\partial v} + \mu_3^+ w_1 \frac{\partial}{\partial w_1},$$

where the vector $\mu_\pm = (\mu_1^\pm, \mu_2^\pm, \mu_3^\pm)$ satisfies the following equations

$$\langle (\mu_1^\pm, \mu_2^\pm, \mu_3^\pm), (a, \epsilon_\pm, 0) \rangle = 0, \quad \langle (\mu_1^\pm, \mu_2^\pm, \mu_3^\pm), (1, 0, 1) \rangle = 0.$$

Here \langle, \rangle is the usual scalar product on \mathbb{C}^3 . By simple calculations, we obtain

$$X_\pm(u_1, v_\pm, w_1) = \pm \epsilon_\pm u_1 \frac{\partial}{\partial u_1} \mp a v_\pm \frac{\partial}{\partial v_\pm} \mp \epsilon_\pm w_1 \frac{\partial}{\partial w_1}.$$

Similar computation shows that there are local coordinates near q_{\pm} in which the vector field generating the foliation is given by

$$X_{\pm}(u_{\pm}, v_{\pm}, w_{\pm}) = -\epsilon w_{\pm} \frac{\partial}{\partial v_{\pm}} + \epsilon_{\pm} w_{\pm} \frac{\partial}{\partial w_{\pm}}.$$

□

3 Proof of the theorem

Let $t = \frac{\lambda^a}{h}$. The blown-up foliation $\sigma_1^* \mathcal{F}$ is given by the two first integrals

$$\begin{aligned} G &= w_1^a (1 - w_1)^{-\epsilon} (v_1 - 1)^{-\epsilon_+} (v_1 + 1)^{-\epsilon_-} \tilde{\Delta}^{-1} = t, \\ F &= u_1 w_1 = \lambda. \end{aligned}$$

Consider the two-dimensional square $Q \subset \mathbb{CP}^2$ with vertices p_+, p_-, q_+ and q_- . All levels curves $\{G = t\}$ inside Q correspond to values of $t \in [0, +\infty]$. We consider the family of hyperbolic polycycles δ^t , i.e. at each intersection of two consecutive curve we have a saddle point,.

$$\delta^t = (\sigma_1^{-1}(\gamma(0, 0) \setminus (0, 0, 0)) \cup (Q \cap \{G = t\}))^{\mathbb{R}}, t \in [0, +\infty].$$

where $(\dots)^{\mathbb{R}}$ denotes the real part of a complex analytic set. All polycycle δ^t satisfies the genericity assumptions from [2].

Let $\delta(\lambda, t) = \sigma^{-1}(\gamma(\lambda, h)) \subset W$ be the pull-back of the cycle $\gamma(\lambda, h)$ by the blowing-up map. Let δ^t be the polycycle corresponding to the cycle $\delta(\lambda, t)$.

Let

$$J(\lambda, t) = \int_{\delta(\lambda, t)} \sigma_1^* \frac{\eta}{M_{\lambda}}.$$

The integral $J(\lambda, t)$ can be analytically continued to the universal cover of $\mathbb{C}^2 \setminus \{\lambda t = 0\}$.

3.1 Variation operator

Given any multivalued function F defined in a neighborhood of the origin in \mathbb{C} i.e. a holomorphic function defined on the universal covering $\widetilde{\mathbb{C}^*}$ of \mathbb{C}^* . We define the rescaled monodromy as

$$\mathcal{Mon}_{(t, \alpha)} F(t) = F(te^{i\pi\alpha}).$$

The variation is given as the difference between the counterclockwise and clockwise continuation

$$\begin{aligned} \mathcal{Var}_{(t, \alpha)} F(t) &= \mathcal{Mon}_{(t, \alpha)} J(t) - \mathcal{Mon}_{(t, -\alpha)} J(t) \\ &= F(te^{i\pi\alpha}) - F(te^{-i\pi\alpha}). \end{aligned}$$

Now, let G be a multivalued function in two variables λ and t defined in universal covering $\mathbb{C}^2 \setminus \widetilde{\{st = 0\}}$ of $\mathbb{C}^2 \setminus \{st = 0\}$. We define the mixed variation as

$$\begin{aligned} \mathcal{Var}_{(\beta, \lambda)} \circ \mathcal{Var}_{\alpha, t} G(\lambda, t) &= \mathcal{Var}_{(\beta, \lambda)} (G(\lambda, te^{i\pi\alpha}) - G(\lambda, te^{-i\pi\alpha})) = \\ &= G(\lambda e^{i\pi\beta}, te^{i\pi\alpha}) - G(\lambda e^{-i\pi\beta}, te^{i\pi\alpha}) - G(\lambda e^{i\pi\beta}, te^{-i\pi\alpha}) + G(\lambda e^{-i\pi\beta}, te^{-i\pi\alpha}). \end{aligned}$$

Lemma 1 *The variations $\mathcal{Var}_{(\lambda, \beta)}$ and $\mathcal{Var}_{(t, \alpha)}$ commute*

$$\mathcal{Var}_{(\lambda, \beta)} \circ \mathcal{Var}_{(t, \alpha)} = \mathcal{Var}_{(t, \alpha)} \circ \mathcal{Var}_{(\lambda, \beta)}.$$

Proof. The proof is a direct consequence of monodromy theorem. □

3.2 The analytic properties of the integral J near the polycycle δ^t

Using the partition of unity of the blown-up space we can decompose our cycle of integration $\delta(\lambda, t)$ in a relative cycles and we can check that any relative cycle can be chosen as a lift of a base path (for more details see [3]). Then, we have

Proposition 3 . *The integral $J(\lambda, t)$ satisfies the rescaled iterated variation equations*

$$\mathcal{V}ar_{(t, \beta_1)} \circ \dots \circ \mathcal{V}ar_{(t, \beta_k)} J(\lambda, t) = 0, \quad (7)$$

$$\mathcal{V}ar_{(\lambda, 1)} \circ \mathcal{V}ar_{(\lambda, 1)} J(\lambda, t) = 0, \quad (8)$$

where β_i are analytic functions in $\epsilon, \epsilon_+, \epsilon_-, \dots, \epsilon_k$.

3.3 Increment of argument of J

The estimation of a local bound of the number of isolated zeros of the pseudo-abelian integral $I(\lambda, h) = \int_{\gamma(\lambda, h)} \frac{\eta}{M_\lambda}$ is analogous to estimate a local λ -uniform bound of isolated zeros of the integral $J(\lambda, t) = \int_{\delta(\lambda, t)} \sigma_1^* \frac{\eta}{M_\lambda}$. As consequence of the equations (7) and (8) the integral $J(s, t)$ has the following expansion

$$J(\lambda, t) = J_1(\lambda, t) + J_2(\lambda, t) \log \lambda, \quad (9)$$

where

$$J_2(\lambda, t) = \mathcal{V}ar_{(\lambda, 1)} J(\lambda, t) = \int_{\text{eight loop}} \sigma_1^* \frac{\eta}{M_\lambda}, \quad (10)$$

$$\mathcal{V}ar_{(t, \beta_1)} \circ \dots \circ \mathcal{V}ar_{(t, \beta_k)} J_i(\lambda, t) = 0, \quad i = 1, 2, \quad (11)$$

$$\mathcal{V}ar_{(\lambda, 1)} J_i(\lambda, t) = 0, i = 1, 2. \quad (12)$$

Theorem 2. *Let $\epsilon > 0$ be sufficiently small. Then, for all $|\lambda| < \epsilon$ the number of zeros*

$$\#\{t \in [0, +\infty] : J(\lambda, t) = 0\}$$

is uniformly bounded with respect to λ .

The integral $J(\lambda, t)$ has analytic prolongation to the complex argument t . This is a multivalued function with unique ramification point $t = 0$.

Let $\partial\Omega$ be the boundary of a domain complex Ω which consists of a big circular arc $C_{R_1} = \{|t| = R_1, |\arg t| \leq \alpha\pi\}$, a two segments $C^\pm = \{r_1 \leq |t| \leq R_1, |\arg t| = \pm\alpha\pi\}$ and the small circular arc $C_{r_1} = \{|t| = r_1, |\arg t| \leq \alpha\pi\}$.

To count the number of zeros of the function $J(s, t)$, we estimate the increment of argument of the function $J(\lambda, t)$ we apply the argument principle which says that

$$\#Z(J_\Omega) \leq \frac{1}{2\pi} \Delta \arg(J_{\partial\Omega}) = \frac{1}{2\pi} (\Delta \arg(J_{C_{R_1}}) + \Delta \arg(J_{C_{r_1}}) + \Delta \arg(J_{C^\pm}))$$

The increment of arguments $\Delta \arg(J_{C_{R_1}})$ and $\Delta \arg(J_{C^\pm})$ are locally uniformly bounded, for more details see [3]. The problem consist to estimate the increment of argument of the integral $J(s, t)$ along the small circular arc C_{r_1} .

Let

$$\Lambda(\lambda; n_0, n_1, \dots, n_k; n) = \{M_\lambda \frac{dH_\lambda}{H_\lambda} + \kappa(Rdx + Rdy) : H_\lambda = P_\lambda \prod_{i=1}^k P_i^{\epsilon_i}, \\ \deg P_\lambda \leq n_0, \deg P_i \leq n_i, \deg(R, S) \leq n\}.$$

be the parameters space. Consider the following functional space \mathcal{P}

$$\mathcal{P}(v, V; \alpha_1, \dots, \alpha_k; \lambda) := \left\{ \sum \sum c_{jln}(\lambda) t^{\alpha_j n} \log^n t, c_{jln} \in \mathbb{C}, v \leq \alpha_j n \leq V, 0 \leq l \leq k \right\}.$$

3.3.1 Finite order approximation to $J(\lambda, t)$

Our goal is to obtain an asymptotic expansion for $J(\lambda, t)$. This will allow us to prove the existence of an upper bound for the increment of the argument $\Delta \arg(J_{C_{r_1}})$. The problem is proposed in the fact that the existence of the term $\log(\lambda) \rightarrow \infty$ as $\lambda \rightarrow 0$ in the expression of the function $J(\lambda, t)$. To resolve the problem we make a weighted quasi-homogeneous blowing-up. Let

$$c = (\log \lambda)^{-1}, \quad a = J_1(\lambda, t), \quad b = J_2(\lambda, t).$$

Then, we have

$$J(\lambda, t) = J_1(\lambda, t) + \log(\lambda)J_2(\lambda, t) = c^{-1}(ca + b) = c^{-1}\psi(a, b, c).$$

As $c^{-1} = \log \lambda \in \mathbb{R}$, we have

$$\arg(J(\lambda, t)) = \arg(c^{-1}\psi(a, b, c)) = \arg(\psi(a, b, c)).$$

Proposition 4. *There exist a some meromorphic function ϕ such that*

$$\Delta \arg(\psi(a, b, c)) = \Delta \arg(\phi).$$

Proof. To estimate the increment of argument $\arg(q(a, b, c))$ of polynomial q we make a quasi-homogeneous blowing-up of weight $(\frac{1}{2}, 1, \frac{1}{2})$ with the center $C_1 = \{a = b = c = 0\}$. The explicit formulae of the blowing-up π_1^{-1} in the affine charts $\tau_1 = \{a \neq 0\}$, $\tau_2 = \{b \neq 0\}$ and $\tau_3 = \{c \neq 0\}$ is written respectively as

$$\begin{aligned} \pi_1^{-1}|_{\tau_1} &= \pi_{11}^{-1} : a = \sqrt{a_1}, \quad b = b_1 a_1, \quad c = c_1 \sqrt{a_1}, \\ \pi_1^{-1}|_{\tau_2} &= \pi_{12}^{-1} : a = a_2 \sqrt{b_2}, \quad b = b_2, \quad c = c_2 \sqrt{b_2}, \\ \pi_1^{-1}|_{\tau_3} &= \pi_{13}^{-1} : a = a_3 \sqrt{c_3}, \quad b = b_3 c_3, \quad c = \sqrt{c_3}. \end{aligned}$$

The total transform $\pi_1^* q(a, b, c)$ of $q(a, b, c)$ is given, in different charts, by

$$\begin{aligned} \pi_{11}^*(ca + b) &= a_1(c_1 + b_1) = a_1 P_1(a_1, b_1, c_1), \\ \pi_{12}^*(ca + b) &= b_2(a_2 c_2 + 1) = b_2 P_2(a_2, b_2, c_2), \\ \pi_{13}^*(ca + b) &= c_3(a_3 + b_3) = c_3 P_3(a_3, b_3, c_3). \end{aligned}$$

where $\{a_1 = 0\}$, $\{b_2 = 0\}$ and $\{c_3 = 0\}$ are local equations of the exceptional divisor and $\{P_1 = 0\}$, $\{P_2 = 0\}$ and $\{P_3 = 0\}$ local equations of strict transform of $\{ca + b = 0\}$.

We observe that the exceptional divisor $\{a_1 = 0\}$ (resp $\{c_3 = 0\}$) has not a normal crossing with the strict transform $\{P_1 = 0\}$ (resp $\{P_3 = 0\}$). Conversely in the chart τ_2 , the exceptional divisor $\{b_2 = 0\}$ has a normal crossing with the strict transform $\{P_2 = 0\}$. To resolve this problem we make a second blowing-up π_2^{-1} with center C_2 such that

1. in the chart τ_1 with coordinates (a_1, b_1, c_1) , the center C_2 is given by $C_2 = \{c_1 = b_1 = 0\}$,
2. in the chart τ_2 the blowing-up π_2 is a biholomorphism,
3. in the chart τ_3 with coordinates (a_3, b_3, c_3) , the center C_2 is given by $C_2 = \{a_3 = b_3 = 0\}$,

and

1. in the chart τ_1 , we have $\pi_2^{-1}(C_2)$ is covered by two coordinates charts V_{b_1} and V_{c_1} with coordinate $(\tilde{a}_1, \tilde{b}_1, \tilde{c}_1)$ such that the blowing-up π_2^{-1} is given in the charts V_{b_1} and V_{c_1} respectively, by $a_1 = \tilde{a}_1, b_1 = \tilde{b}_1, c_1 = \tilde{b}_1 \tilde{c}_1$ and $a_1 = \tilde{a}_1, b_1 = \tilde{b}_1 \tilde{c}_1, c_1 = \tilde{c}_1$,

2. in the chart τ_3 , we have $\pi_2^{-1}(C_2)$ is covered by two coordinates charts V_{a_3} and V_{b_3} with coordinates $(\tilde{a}_3, \tilde{b}_3, \tilde{c}_3)$ such that the blowing-up π_2^{-1} is given in V_{a_3} and V_{b_3} respectively, by $a_3 = \tilde{a}_3, b_3 = \tilde{a}_3\tilde{b}_3, c_3 = \tilde{c}_3$ and $a_3 = \tilde{a}_3\tilde{b}_3, b_3 = \tilde{a}_3, c_3 = \tilde{c}_3$.

Then, we have

1. in the chart V_{b_1} ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{a}_1 \tilde{b}_1 = J_2(\lambda, t),$$

2. in the chart V_{c_1} ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{a}_1 \tilde{c}_1 = \frac{J_1(\lambda, t)}{\log \lambda},$$

3. in the chart V_{a_3} ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{a}_3 \tilde{c}_3 = \frac{J_1(\lambda, t)}{\log \lambda},$$

4. in the chart V_{b_3} ,

$$\pi_2^* \circ \pi_1^* \psi(a, b, c) \stackrel{0}{\approx} \tilde{b}_3 \tilde{c}_3 = \frac{J_1(\lambda, t)}{\log \lambda},$$

5. in the chart τ_2 , the exceptional divisor $\{b_2 = 0\}$ has a normal crossings with the strict transform $P_2 = 0$ and the blowing-up π_2^{-1} is a biholomorphism, so the function

$$\frac{J_1(\lambda, t)}{\log \lambda} + J_2(\lambda, t) = \frac{J(\lambda, t)}{\log \lambda}$$

is meromorphic function.

Then, we conclude that $\phi \in \left\{ \frac{J(\lambda, t)}{\log \lambda}, \frac{J_1(\lambda, t)}{\log \lambda}, J_2(\lambda, t) \right\}$. □

3.3.2 Proof of the theorem 2

Using lemma 4.8 of [2] for $\phi \in \left\{ \frac{J_1(\lambda, t)}{\log \lambda}, J_2(\lambda, t) \right\}$, there exist a $\tilde{\phi} \in \mathcal{P}(\dots)$ such that $|\phi(\lambda, t) - \tilde{\phi}(\lambda, t)| \leq N$. For $\phi = \frac{J(\lambda, t)}{\log \lambda}$, we can conclude using Gabrielov's theorem [4].

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